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The short-range-interaction $\pm J$-random-bond spherical quantum spin glass on the Bethe lattice: dynamic correlations and thermodynamic functions

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Abstract. We consider the short-range-interaction disordered quantum spherical model with a symmetrical binary $\pm J$-bond distribution on the Bethe lattice (with coordination number z). The system exhibits a quantum phase transition separating the spin-glass and disordered phases, where the quantum effects are regulated by a parameter Δ describing the kinetic energy. The model admits, in the spherical limit, an exact solution both in the spin-glass phase and in the disordered phase. The quantum dynamics is examined via various correlation functions on the infinite tree, which are evaluated in closed form. A closed-form solution for the averaged free energy has been obtained, and various thermodynamic functions are explicitly derived in the form of low-temperature expansions.

1. Introduction

Systems exhibiting glassy phases constitute the main attempt of solid-state physics to address the problem of collective disorder. The significance of these systems stems not only from the importance of understanding particular materials; rather, it is believed that the glassy ordering is of a qualitatively new kind, prototypical for a variety of disordered materials. *Classical* systems exhibiting spin-glass behaviour have been intensively investigated [1], and it has proved extremely useful to consider the *infinite-range* Sherrington–Kirkpatrick (SK) model [2]. In the classical case, it is generally agreed that the statistical mechanics of the SK Ising spin glass is essentially understood: the replica-symmetry-breaking (RSB) mean-field solution due to Parisi [3] exhibits infinitely many low-temperature states whose properties are consistent with simulations [4] (see reference [5] for a recent overview).

Recently, however, there has been growing experimental and theoretical interest in the properties of *quantum* disordered systems. These include spin-glass (SG) problems and other quantum transitions such as the metal–insulator [6] and the Bose glass [7] transitions. In quantum spin glasses the spin-glass-ordered ground state may be destabilized by quantum fluctuations, e.g. by varying the strength of the quantum fluctuations—for example, by changing the concentrations of magnetic spins or itinerant electrons in magnetic systems or applying a transverse magnetic field in an Ising dipolar system [8] $LiHo_xY_{1-x}F_4$. The theoretical problems for the *quantum* system are much less well understood, and the problem of determining the equilibrium ground-state structure of quantum spin glasses remains an important question.

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Much work in the past has been confined to a large number of *infinite-range*-interaction quantum spin glasses, where each spin interacts with every other spin. A celebrated example is the Sherrington–Kirkpatrick (SK) spin-glass model [9] in a transverse field [10]. For *short-range*-interaction *quantum* SG systems, very little information has been established analytically, and what has been is mainly for one-dimensional quantum Ising chains [11], via the phenomenological theory of droplet excitations [12] and by the renormalization group (RG) method [13] (though with limited application for physically interesting dimensionalities $d < d_c = 8$ due to runaway RG flow to strong coupling for $d < d_c$). Virtually all of the available data for realistic short-range quantum SG models result from extensive Monte Carlo simulations performed for two-dimensional [14] and three-dimensional [15] transverse-field Ising systems.

Complementary to the numerical simulations are the fully solvable SG models, whose importance lies in the fact that they can provide valuable insights and controlled guides for developing our understanding as to what can happen in real systems. In this context, an intriguing example is given by the Bethe lattice—i.e. an infinite homogeneous hierarchical structure that greatly simplifies a number of problems of statistical physics.

Statistical models on the Bethe lattice have attracted considerable interest for a long time because they admit a direct analytical approach for a number of problems that are otherwise intractable on Euclidean lattices. In general, the study of a variety of problems on the Bethe lattice has helped to develop our understanding of a number of physical issues including self-avoiding polymers [16], random resistor networks [17], percolation [18], and classical spin glasses on random graphs (see reference [19]) to mention but a few. The physical relevance of these results is that the Bethe lattice is thought to represent some mean-field limit of Euclidean lattices of very large dimensions.

In spite of there having been a number of papers on the mean-field SK-type models of quantum spin glasses, the connection between the mean-field models and short-range models remains obscure, partially due to the absence of a notion of distance in the infinite-range-interaction Sherrington–Kirkpatrick model, which makes it impossible to define non-trivial site-dependent correlation functions. In contrast, the coordination number of a Bethe lattice is finite, there is an obvious notion of distance as measured along the tree, and one can thus consider correlation functions as a function of distance, at least by rough analogy with short-range systems.

In this paper we report analytic results for the *quantum* realization of the *short-range*-interaction spin glass in the form of the disordered $\pm J$ quantized spherical model formulated on the Bethe lattice with constant coordination z. We define genuine nearest-neighbour random couplings, resulting in non-trivial correlations between spins. We concentrate our attention on the important aspect of the quantum SG transition: various inherent *dynamical* correlations like the local response and higher-order site-dependent dynamic spin correlations, which typically signal SG transition, are evaluated in closed form. Furthermore, we examine in detail the thermodynamic properties of the model. Some preliminary results on these issues were reported in our recent work [20].

The outline of the remainder of the paper is as follows. In section 2 we begin by setting up the spherical model of the quantum SG on the Bethe lattice and the corresponding Euclidean action; this functional is minimized to yield the mean-field theory. In section 3 the SG order parameter and various susceptibilities are discussed. We show that in the formal limit where the coordination number of the lattice tends to infinity, the solution of the model becomes that of the infinite-range-interaction SK-type spherical SG. The thermodynamics of the model is developed in section 4 in terms of low-temperature expansions for various thermodynamic functions. Finally, in section 5 we summarize the conclusions to be drawn from our work.

Some supplementary material regarding the connection between spherical and large-M vector models and the classical limit of the spherical SG models appear in the appendices.

2. The model Hamiltonian

To capture the essential physics of the problem, we consider a quantized spherical model on the Bethe lattice given by the Hamiltonian

$$H = \frac{\Delta}{2}\sum_i \Pi_i^2 - \sum_{i<j} J_{ij}\sigma_i\sigma_j \qquad (1)$$

where the variables σ_i ($i = 1, \ldots, N$) are associated with spin degrees of freedom (located on the Bethe lattice with coordination z) and canonically conjugated to the 'momentum' operators Π_i such that $[\sigma_i, \Pi_j] = \mathrm{i}\delta_{ij}$. In analogy to the transverse-field Ising SG, the coupling Δ regulates the strength of the quantum fluctuations ($\Delta \to 0$ corresponds to the classical limit). Furthermore, the J_{ij} are mutually uncorrelated nearest-neighbour exchange constants which we assume to be selected with the probability

$$P(J_{ij}) = \frac{1}{2}\left[\delta(J_{ij} - J) + \delta(J_{ij} + J)\right]. \qquad (2)$$

Finally, we supplement equation (1) with the mean-spherical constraint

$$\left\langle\left\langle N^{-1}\sum_{i=1}^{N}\sigma_i^2\right\rangle_T\right\rangle_{\mathrm{av}} = 1 \qquad (3)$$

where $\langle\cdots\rangle_T$ and $\langle\cdots\rangle_{\mathrm{av}}$ denote the ensemble and random averages, respectively.

2.1. *Euclidean action and the saddle-point condition*

To establish the framework for our analytical task, we express the partition function $Z = \mathrm{Tr}\,\mathrm{e}^{-H/k_BT}$ using the functional integral in the Matsubara 'imaginary-time' τ-formulation ($0 \leqslant \tau \leqslant 1/k_BT \equiv \beta$, with T being the temperature). We obtain

$$f = -\lim_{N\to\infty}\frac{1}{\beta N}\langle\ln Z_N(\{J_{ij}\})\rangle_{\mathrm{av}} \qquad (4)$$

where

$$Z_N(\{J_{ij}\}) = \int\prod_i[\mathrm{D}\sigma_i(\tau)]\,\delta\left(\sum_{i=1}^{N}\sigma_i^2(\tau) - N\right)\exp\left(-\int_0^\beta \mathrm{d}\tau\; S_\sigma(\tau)\right) \qquad (5)$$

with the Euclidean action

$$S_\sigma(\tau) = \frac{1}{2\Delta}\sum_i\left(\frac{\partial\sigma_i}{\partial\tau}\right)^2 - \sum_{i<j}J_{ij}\sigma_i(\tau)\sigma_j(\tau). \qquad (6)$$

The convenient way to enforce the spherical constraint is to use the functional analogue of the δ-function representation:

$$\delta\left(\sum_{i=1}^{N}\sigma_i^2(\tau) - N\right) = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\frac{\mathrm{D}v(\tau)}{2\pi\mathrm{i}}\right]\exp\left[\int_0^\beta \mathrm{d}\tau\; v(\tau)\left(\sum_{i=1}^{N}\sigma_i^2(\tau) - N\right)\right] \qquad (7)$$

so that we obtain

$$Z_N(\{J_{ij}\}) = \int\prod_i[\mathrm{D}\sigma_i(\tau)]\;\exp\left(-\int_0^\beta \mathrm{d}\tau\; Nv(\tau)\right)\Xi_N(\beta,\{J_{ij}\},v) \qquad (8)$$

where

$$\Xi_N(\beta, \{J_{ij}\}, v) = \exp\left\{-\int_0^\beta \mathrm{d}\tau \left[S_\sigma(\tau) - v(\tau)\sum_{i=1}^N \sigma_i^2(\tau)\right]\right\} \tag{9}$$

which introduces the Lagrange multiplier $v(\tau)$, adding an additional quadratic term (in σ-fields) to the action (6) and allowing us to perform N independent traces over σ_i. In the thermodynamic limit $N \to \infty$, the method of steepest descents is exact and the saddle point $v(\tau) \equiv v_0$ (independent of τ) will satisfy

$$1 = \lim_{N\to\infty} \frac{\partial}{\partial v} \langle \ln \Xi_N(\beta, \{J_{ij}\}, v)\rangle_{\mathrm{av}}\Big|_{v(\tau)\equiv v_0} \tag{10}$$

or explicitly

$$1 = \lim_{N\to\infty} \beta^{-1} \sum_\ell \left\langle \left[\frac{1}{(\omega_\ell^2/\Delta + 2v_0)\mathbf{1}_N - \mathbf{J}}\right]_{ii}\right\rangle_{\mathrm{av}} \tag{11}$$

with $\omega_\ell = 2\pi\ell/\beta$ ($\ell = 0, \pm 1, \pm 2, \ldots$) being the (Bose) Matsubara frequencies. Thus the spherical limit for dirty phase transition produces a random-matrix inversion problem [21] which, in general, is not analytically tractable, forcing numerical implementation [22]. Fortunately, the Bethe lattice topology provides a unique setting that makes explicit progress possible. To proceed, we diagonalize the random symmetric matrix J_{ij} for N spins

$$\sum_i J_{ij}\phi_j^\lambda = J_\lambda \phi_j^\lambda \tag{12}$$

with the real orthonormal eigenvectors ϕ_i^λ. Here, $\lambda = 1, \ldots, N$ and J_λ is the λth eigenvalue. Associated with the eigenvalue spectrum is the averaged integrated density of states (DOS):

$$\rho(\kappa) = -\frac{1}{\pi} \operatorname{Im}\langle \mathcal{G}(\kappa + \mathrm{i}0^+)\rangle_{\mathrm{av}} \tag{13}$$

where

$$\mathcal{G}(\kappa) = \lim_{N\to\infty} N^{-1} \sum_\lambda (\kappa - J_\lambda)^{-1} \tag{14}$$

is the (unaveraged) Green function for the Bethe lattice leading to the averaged integrated DOS given by [20]

$$\rho(\kappa) = \frac{2(z-1)z\sqrt{J_c^2 - \kappa^2}\,\Theta(1 - \kappa^2/J_c^2)}{\pi[(z-2)^2\kappa^2 + z^2(J_c^2 - \kappa^2)]} \tag{15}$$

where $\Theta(x)$ stands for the unit step function and $J_c = 2J\sqrt{z-1}$ is the upper limit of the eigenvalue spectrum. In the spherical model the Lagrange multiplier $2v_0$ 'sticks' at that value at criticality and stays constant throughout the whole low-temperature phase; also,

$$1 = \mathcal{P}\int_{-J_c}^{J_c} \mathrm{d}\kappa\ \rho(\kappa) \frac{1}{\beta} \sum_{\omega_\ell} \frac{1}{\omega_\ell^2/\Delta + 2v_0 - \kappa} \tag{16}$$

and

$$1 = \frac{\sqrt{\Delta}}{2}\mathcal{P}\int_{-J_c}^{J_c} \mathrm{d}\kappa\ \rho(\kappa) \frac{\coth(\beta\sqrt{\Delta(2v_0 - \kappa)}/2)}{\sqrt{2v_0 - \kappa}} \tag{17}$$

and $\mathcal{P}$ denotes the principal part of the integral.

3. The order parameter and observables

3.1. The Edwards–Anderson order parameter

The quantity of interest which captures the onset of the SG phase transition is the Edwards–Anderson (EA) order parameter

$$q_{\mathrm{EA}} \equiv \langle \langle \sigma_i \rangle_T^2 \rangle_{\mathrm{av}}. \tag{18}$$

Within our quantum spherical description,

$$q_{\mathrm{EA}} = 1 - \mathcal{P} \int_{-J_c}^{J_c} \mathrm{d}\kappa \; \rho(\kappa) \frac{1}{\beta} \sum_{\omega_\ell} \frac{1}{\omega_\ell^2/\Delta + 2v_0 - \kappa} \tag{19}$$

and, summing up Matsubara frequencies,

$$q_{\mathrm{EA}} = 1 - \frac{\sqrt{\Delta}}{2} \mathcal{P} \int_{-J_c}^{J_c} \mathrm{d}\kappa \; \rho(\kappa) \frac{\coth(\beta\sqrt{\Delta(2v_0 - \kappa)}/2)}{\sqrt{2v_0 - \kappa}}. \tag{20}$$

Setting $q_{\mathrm{EA}} = 0$ gives the critical phase boundary (see figure 1).

At the *zero-temperature* quantum critical point, q_{EA} is given by

$$q_{\mathrm{EA}}(T = 0, \Delta) = 1 - \alpha(z)\sqrt{\frac{\Delta}{J}} \tag{21}$$

where

$$\alpha(z) = -\frac{1}{8\pi}\sqrt{z + 2\sqrt{z-1}} \ln\left(\frac{2(z + 2\sqrt{z-1})}{z - 2\sqrt{z-1}} - \frac{4\sqrt[4]{z-1}\sqrt{z + 2\sqrt{z-1}}}{z - 2\sqrt{z-1}} - 1 \right) - \frac{1}{4\pi}\sqrt{z - 2\sqrt{z-1}} \arctan\left(\frac{2\sqrt[4]{z-1}}{\sqrt{z - 2\sqrt{z-1}}} \right). \tag{22}$$

Equation (21) implies the zero-temperature critical value $\Delta_c/J = \alpha^{-2}(z)$. Unlike in the case of the $d = 1$ disordered transverse Ising chain [11], we found no evidence for $T = 0$ phase transition at a finite *non-zero* Δ in one dimension, because $\lim_{z\to 2} \alpha(z) = \infty$. Note that

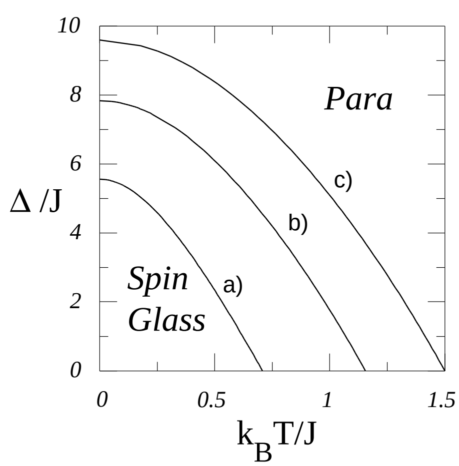

Figure 1. The Δ–T phase diagram of the quantum spherical $\pm J$ model on the Bethe lattice for several coordination numbers z: (a) $z = 3$, (b) $z = 4$ and (c) $z = 5$; the solid lines separate the spin-glass (SG) from the paramagnetic (PM) region.

this conclusion is consistent with recent numerical findings within the disordered quantum spherical model on the two-dimensional regular lattice [22]. In the opposite limit of infinite connectivity ($z \to \infty$), we recover—according to the central-limit theorem—the random Gaussian-distributed infinite-range-interaction spherical spin-glass modes (see appendix A).

3.2. Local susceptibilities

It is also helpful to introduce here a number of correlation functions of the order parameter. Consider now the two-point correlation function

$$g_{ij}(\tau) = \langle \sigma_i(\tau)\sigma_j(0)\rangle_T. \tag{23}$$

The disorder average of this quantity

$$G_{ij}(\tau) = \langle\langle \sigma_i(\tau)\sigma_j(0)\rangle_T\rangle_{\mathrm{av}} \tag{24}$$

vanishes for all pairs ij unless $i = j$ since any configuration of bonds $\{J_{ij}\}$ has a partner $\{J_{ij}\}'$ such that $g_{ij}(\tau, \{J_{ij}\}) = -g_{ij}(\tau, \{J_{ij}\}')$. Therefore, $G_{ij}(\tau) = G(\tau)\delta_{ij}$, and using the Fourier transform

$$G(\tau) = \beta^{-1}\sum_{\ell} \mathrm{e}^{\mathrm{i}\omega_\ell\tau} G(\omega_\ell) \tag{25}$$

we obtain

$$G(\omega_\ell) = \frac{2(z-1)}{J_c^2 z^2 - 4(z-1)(2v_0 + \omega_\ell^2/\Delta)^2} \times \left[(z-2)\left(2v_0 + \frac{\omega_\ell^2}{\Delta}\right) - z\sqrt{\left(2v_0 + \frac{\omega_\ell^2}{\Delta}\right)^2 - J_c^2}\,\right]. \tag{26}$$

The static susceptibility is given by $\chi \equiv G(\omega_\ell = 0)$. For $z \geqslant 3$, $\chi = [2(z-1)/(z-2)]J_c^{-1}$ at criticality and everywhere in the low-temperature phase. The temperature behaviour of the

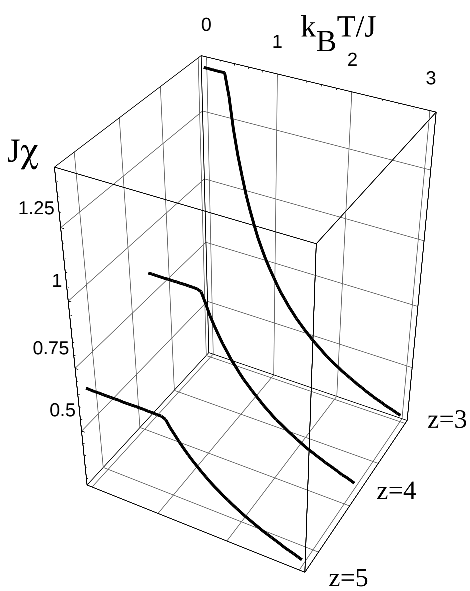

Figure 2. The temperature dependence of the local static susceptibility $\chi = \chi(\omega = 0)$ for $\Delta = 4J$.

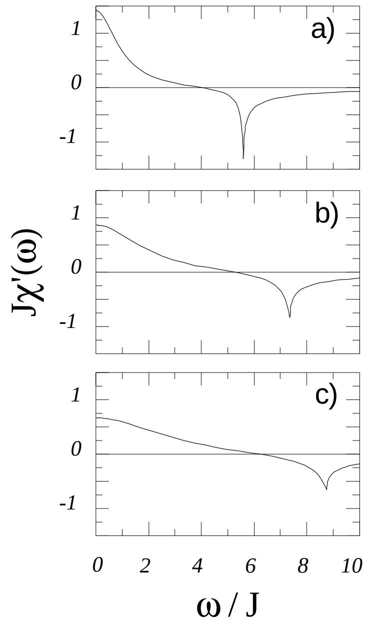

Figure 3. The real part of the dynamic local susceptibility at $\Delta = \Delta_c$ and $T = 0$ for (a) $z = 3$, (b) $z = 4$ and (c) $z = 5$.

static susceptibility is given in figure 2. For the real part of the dynamic local susceptibility $\chi'(\omega)$ we obtain

$$\chi'(\omega) = \begin{cases} \dfrac{2(z-1)\left[(z-2)(2v_0-\omega^2/\Delta) + z\sqrt{(2v_0-\omega^2/\Delta-J_c)(2v_0-\omega^2/\Delta+J_c)}\right]}{J_c^2z^2-4(z-1)(2v_0-\omega^2/\Delta)^2} \\ \qquad \text{for } \omega^2 > \Delta(J_c+2v_0) \\ \dfrac{2(z-1)(z-2)(2v_0-\omega^2/\Delta)}{J_c^2z^2-4(z-1)(2v_0-\omega^2/\Delta)^2} \\ \qquad \text{for } \Delta(2v_0-J_c) \leqslant \omega^2 \leqslant \Delta(J_c+2v_0) \\ \dfrac{2(z-1)\left[(z-2)(2v_0-\omega^2/\Delta) - z\sqrt{(2v_0-\omega^2/\Delta-J_c)(2v_0-\omega^2/\Delta+J_c)}\right]}{J_c^2z^2-4(z-1)(2v_0-\omega^2/\Delta)^2} \\ \qquad \text{for } \omega^2 < \Delta(2v_0-J_c). \end{cases} \tag{27}$$

The frequency dependence of $\chi'(\omega)$ at $T = 0$ and various coordination numbers is given in figure 3. We consider next the dissipative part of the local dynamic susceptibility

$$\chi''(\omega) = \mathrm{Im}[G(\omega_\ell)|_{\mathrm{i}\omega_\ell\to\omega+\mathrm{i}0^+}] \tag{28}$$

which is a quantity often measured in neutron scattering experiments:

$$\chi''(\omega) = -\mathrm{sgn}(\omega)\frac{z}{2}\Theta\left(1-\left|\frac{2v_0}{J_c}-\frac{\omega^2}{\Delta J_c}\right|\right) \times \frac{\sqrt{(-2v_0+\omega^2/\Delta+J_c)(2v_0-\omega^2/\Delta+J_c)}}{(\omega^2/\Delta-2v_0+J_cz/[2\sqrt{z-1}])(\omega^2/\Delta+2v_0-J_cz/[2\sqrt{z-1}])}. \tag{29}$$

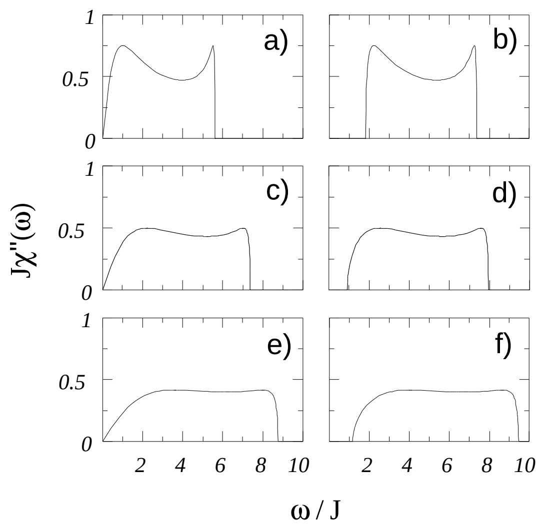

Figure 4. The spectral density $\chi''(\omega)$ at $T = 0$, for $z = 3$: at (a) $\Delta = \Delta_c$ and (b) $\Delta = 9J > \Delta_c$; for $z = 4$: with (c) and (d) corresponding to (a) and (b), respectively; for $z = 5$: at (e) $\Delta = \Delta_c$ and (f) $\Delta = 11J$.

The resulting energy spectrum is gapless at the critical point and throughout the whole low-temperature phase; the gap develops a form of criticality generic for quantum SGs with non-degenerate local ground states (see figure 4). Our rigorous result shows the absence of the $\omega \to 0$ divergence in $\chi''(\omega)$ at criticality for $z \geqslant 3$ anticipated on the basis of a finite-size scaling argument [22].

3.3. *Spin-glass susceptibility*

Finally, we comment on a quantity intimately related to the SG ordering, i.e. the quantum mechanical *four*-spin correlation function:

$$\begin{aligned}\Gamma_{ij}(\tau_1-\tau_2,\tau_3-\tau_4) &= \left[\langle\sigma_i(\tau_1)\sigma_j(\tau_2)\rangle\langle\sigma_i(\tau_3)\sigma_j(\tau_4)\rangle\right]_{\mathrm{av}} \\ &= \beta^{-2}\sum_{\omega_\ell,\nu_\ell} \mathrm{e}^{\mathrm{i}\omega_\ell(\tau_1-\tau_2)+\mathrm{i}\nu_\ell(\tau_3-\tau_4)}[g_{ij}(\omega_\ell,\{J_{ij}\})g_{ij}(\nu_\ell,\{J_{ij}\})]_{\mathrm{av}}.\end{aligned} \tag{30}$$

The correlator (30) is linked to an important quantity, namely the EA spin-glass susceptibility

$$\chi_s = \sum_{ij}\int_0^\beta \mathrm{d}\tau\ \mathrm{d}\tau'\ \Gamma_{ij}(\tau,\tau') \tag{31}$$

which directly probes the onset of the SG transition. The exact, closed form of the correlation function (30) on the Bethe lattice then becomes [20]

$$\Gamma(\omega_\ell,\nu_\ell) = G(\omega_\ell)G(\nu_\ell)\left[\frac{1+D(\omega_\ell)D(\nu_\ell)}{1-(z-1)D(\omega_\ell)D(\nu_\ell)}\right] \tag{32}$$

where

$$D(\omega_\ell) = \frac{1}{J_c\sqrt{z-1}}\left[\left(2v_0+\frac{\omega_\ell^2}{\Delta}\right)\sqrt{\left(2v_0+\frac{\omega_\ell^2}{\Delta}\right)^2 - J_c^2}\right]. \tag{33}$$

Hence, from equation (32) and the saddle-point condition, it follows that the static SG susceptibility $\chi_s \equiv \Gamma(\omega_\ell = 0, \nu_\ell = 0)$ diverges at the critical point as well as throughout the whole low-temperature phase. Close to the zero-temperature critical point the imaginary part of the dynamic SG susceptibility $\chi_s''(\omega) \equiv \mathrm{Im}[\Gamma(\omega_\ell, \omega_\ell)|_{\mathrm{i}\omega_\ell \to \omega + \mathrm{i}0^+}]$ diverges as $\chi_s''(\omega) \sim 1/\omega$. The behaviour of the dissipative part of the dynamic SG susceptibility is given in figure 5:

$$\chi_s''(\omega, \omega) = \mathrm{sgn}(\omega)\Theta\left(1 - \left|\frac{2v_0}{J_c} - \frac{\omega^2}{\Delta J_c}\right|\right) \times \frac{2(z-1)z(2v_0 - \omega^2/\Delta)\left[4(z-1)(2v_0 - \omega^2/\Delta)^2 + J_c^2 z^2 - 8J_c^2(z-1)\right]}{\sqrt{J_c^2 - (2v_0 - \omega^2/\Delta)^2}\left[J_c^2 z^2 - 4(z-1)(2v_0 - \omega^2/\Delta)^2\right]^2}. \tag{34}$$

Consider now the situation in the vicinity of the zero-temperature paramagnetic–SG transition. Raising the temperature at $\Delta = \Delta_c(T = 0)$ one enters the QC regime in which the physics is dominated by the $T = 0$ quantum critical point. In this regime, for the dissipative part of the dynamic SG susceptibility we obtain

$$\chi_s''(T, \Delta_c(T = 0)) \sim \frac{1}{T}\left[\ln\left(\frac{\text{constant}}{T}\right)\right]^{-1/2}. \tag{35}$$

Here the temperature is the most significant energy scale and the system 'feels' the *finite* value of T before becoming sensitive to the deviation of Δ from $\Delta_c(T = 0)$ [13].

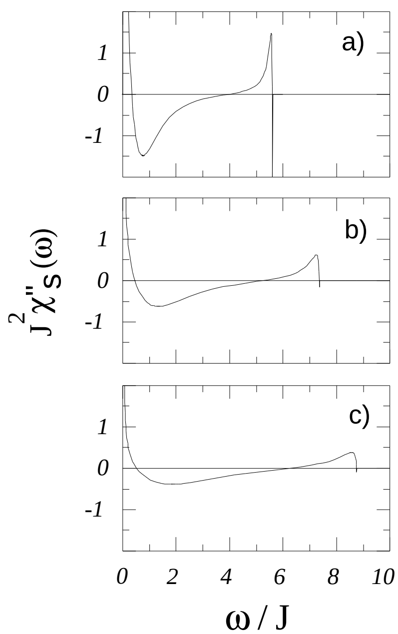

Figure 5. The imaginary part of the dynamic SG susceptibility $\chi_s''(\omega)$ at $\Delta = \Delta_c$ and $T = 0$ for (a) $z = 3$, (b) $z = 4$ and (c) $z = 5$.

4. Thermodynamic functions

4.1. The free energy

Returning to the partition function represented by equation (5) it is evident that the free energy will be given by the dominant saddle-point solution (11) in the thermodynamic ($N \to \infty$) and spherical limits, allowing the exact evaluation of the disorder-averaged free-energy density $f_{\rm av} = [F/MN]_{\rm av}$ where

$$F = -\frac{1}{\beta}\left\langle \ln \det_N \left\{ \left[\left(-\frac{1}{\Delta}\frac{\partial^2}{\partial\tau^2} + 2v_0 \right) \mathbf{1}_N - \mathbf{J} \right] \Big/ \left[\left(-\frac{1}{\Delta}\frac{\partial^2}{\partial\tau^2} \right) \mathbf{1}_N \right] \right\} \right\rangle_{\rm av} - v_0. \tag{36}$$

Proceeding along similar lines to section 2, one finds the exact form of the averaged free energy in the spherical limit:

$$f = \frac{1}{\beta}\mathcal{P}\int_{-J_c}^{J_c} {\rm d}\kappa\ \rho(\kappa) \ln\left[2\sinh\left(\frac{1}{2}\beta\sqrt{2\Delta v_0 - \kappa\Delta} \right) \right] - v_0 \tag{37}$$

with $2v_0 = J_c$ in the SG phase. We are now in position to calculate various thermodynamic functions within the spin-glass phase. The classical case of vanishing Δ is discussed separately, in appendix B.

4.2. The entropy

In any theoretical description of spin glasses the entropy defined by

$$S = k_B\beta^2\frac{\partial f}{\partial\beta} \tag{38}$$

represents the thermodynamic quantity of primary interest. Using relation (37) one gets

$$S = \frac{16k_B}{\pi}\int_0^\mu {\rm d}x\ \frac{(z-1)z\sqrt{\mu^2-x^2}\,[x\coth x - \ln(2\sinh x)]}{\mu^4(z-2)^2 - 16x^2(x-\mu)(x+\mu)(z-1)} \tag{39}$$

where we used the fact that $v_0 = J =$ constant in the spin-glass phase and $\mu = \beta\sqrt{J\Delta}$ is a dimensionless parameter introduced in order to allow us to conveniently analyse various physical limits. The low-temperature behaviour ($\mu \to \infty$) of the entropy in the quantum case is described by the low-temperature expansion

$$S = \frac{8\pi^3(z-1)zA_1(z)}{45\mu^3(z-2)^2} - \frac{4\pi^5(z-1)zA_2(z)}{105\mu^5(z-2)^4} - \frac{\pi^7(z-1)zA_3(z)}{105\mu^7(z-2)^6} - \frac{5\pi^9(z-1)zA_4(z)}{594\mu^9(z-2)^8} - \frac{691\pi^{11}(z-1)zA_5(z)}{240\,240\mu^{11}(z-2)^{10}} + {\rm O}\left(\frac{1}{\mu^{13}}\right) \tag{40}$$

where the coefficients $A_k(z)$ are given in table 1. The relation (40) implies that for $\Delta \neq 0$ the entropy vanishes as T^3 with a positive coefficient.

4.3. The specific heat

The specific heat at constant volume is defined as

$$C = -k_B\beta^2\frac{\partial^2}{\partial\beta^2}(\beta f). \tag{41}$$

Performing the differentiation we obtain

$$C = \frac{16k_B}{\pi}\int_0^\mu {\rm d}x\ \frac{(z-1)zx^4\sqrt{\mu^2-x^2}\,\sinh^{-2}(x)}{\mu^4(z-2)^2 - 16x^2(x-\mu)(x+\mu)(z-1)}. \tag{42}$$

In the low-temperature limit ($\mu \to \infty$) the following expansion holds:

$$C/k_B = \frac{8\pi^3(z-1)zA_1(z)}{15\mu^3(z-2)^2} - \frac{4\pi^5(z-1)zA_2(z)}{21\mu^5(z-2)^4} - \frac{\pi^7(z-1)zA_3(z)}{15\mu^7(z-2)^6} - \frac{5\pi^9(z-1)zA_4(z)}{66\mu^9(z-2)^8} - \frac{691\pi^{11}(z-1)zA_5(z)}{21\,840\mu^{11}(z-2)^{10}} + {\rm O}\left(\frac{1}{\mu^{13}}\right) \tag{43}$$

Table 1. Coefficients for the low-temperature expansion of thermodynamic functions on the Bethe lattice.

k	$A_k(z)$
1	1
2	$z^2 - 36z + 36$
3	$z^4 + 184z^3 - 2984z^2 + 5600z - 2800$
4	$z^6 - 108z^5 + 11\,164z^4 - 130\,208z^3 + 335\,344z^2 - 324\,288z + 10\,8096$
5	$5z^8 - 208z^7 - 59\,216z^6 + 2437\,952z^5 - 23\,718\,944z^4 + 73\,766\,144z^3 - 102\,532\,352z^2 + 66\,808\,832z - 16\,702\,208$

which implies that the specific heat vanishes as T^3 for $T \to 0$. This is in contrast to the classical case of the classical spherical spin glass, where C remains constant as the temperature vanishes [24].

4.4. The internal energy

For the internal energy defined as

$$U = \frac{\partial}{\partial \beta}(\beta f) \tag{44}$$

we obtain the general relation

$$U = \frac{16 k_B T}{\pi} \int_0^{\mu} \mathrm{d}x \, \frac{(z-1)z\sqrt{\mu^2 - x^2}\,x^3 \coth x}{\mu^4 (z-2)^2 - 16x^2 (x-\mu)(x+\mu)(z-1)} \tag{45}$$

and in the limit $\mu \to \infty$ the internal energy is described by

$$\beta U = \frac{2\pi^3 (z-1) z A_1(z)}{15\mu^3 (z-2)^2} - \frac{2\pi^5 (z-1) z A_2(z)}{63\mu^5 (z-2)^4} - \frac{\pi^7 (z-1) z A_3(z)}{120\mu^7 (z-2)^6} - \frac{\pi^9 (z-1) z A_4(z)}{132\mu^9 (z-2)^8} - \frac{691\pi^{11} (z-1) z A_5(z)}{262\,080\mu^{11} (z-2)^{10}} + \mathrm{O}\left(\frac{1}{\mu^{13}}\right) \tag{46}$$

again exhibiting T^3-behaviour at low temperatures.

5. Conclusions

This paper has introduced and analysed the simplest model quantum SG with short-ranged interactions which exhibits a quantum phase transition: the $\pm J$ spherical quantum spin glass on the Bethe lattice. The model contained only bosonic quantum rotor degrees of freedom and offers the simplest realization of the combined effects of randomness and quantum mechanics. We were able to work at any value of the temperature, including $T = 0$, without resorting to approximations. Basic quantities encoding the quantum SG dynamics and ordering, such as the order parameter, and dynamic two-point and higher-order susceptibilities, have been evaluated exactly. The results are summarized in the phase diagram and the figures (figures 1–5). Unlike in the case for the $d = 1$ disordered transverse Ising chain, we found no evidence for a zero-temperature phase transition—a conclusion which is in agreement with recent numerical findings for the disordered quantum spherical model on the two-dimensional regular lattice. In contrast, in the limit of infinite coordination we recover the usual mean-field results for a spherical quantum SG with infinite-range interactions. From our observation that the *local* structures of a Bethe lattice and a physical large-z lattice with *finite d* are similar (compare with the Bethe–Peierls approach), we expect that the behaviour on the infinite tree will share some features with real finite-dimensional quantum SG systems, at least on length scales short compared to the scale at which loops will typically start to form in a random walk on a Euclidean d-dimensional lattice.

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Appendix A. The limit of infinite lattice connectivity

Let us examine our results in the opposite limit of infinite connectivity; substituting $J = \tilde{J}/\sqrt{z}$ and letting $z \to \infty$ in equation (13) we get the limiting distribution

$$\rho_\infty(\epsilon) = (2\pi \tilde{J}^2)^{-1}\sqrt{\epsilon^2 - \tilde{J}^2}\,\Theta(2\tilde{J} - |\epsilon|). \qquad \text{(A.1)}$$

Thus, for a fully connected lattice, we recover the famous semi-circular spectrum of Gaussian-distributed J_{ij} with zero mean and variance $\tilde{J}/\sqrt{N}$ commonly used for the infinite-range SK spin glasses. Because basic results pertinent to these system (both classical and quantum) are known, we can have an independent consistency check of our calculations. From equation (B.9) we obtain

$$q_{\mathrm{EA}} = 1 - k_B T/\tilde{J} \qquad \text{(A.2)}$$

and $k_B T_c/\tilde{J} = 1$ in agreement with the finding for a classical spherical infinite-ranged SG [24]. Similarly, for the quantum case, from equations (21) and (22) we get

$$\lim_{z\to\infty} \frac{1}{\sqrt{z} f^2(z)} = \frac{9\pi^2}{64} \qquad \text{(A.3)}$$

and consequently

$$\Delta_c/\tilde{J} = \lim_{z\to\infty} \alpha^{-2}(z)/\sqrt{z} = 9\pi^2/16. \qquad \text{(A.4)}$$

We thus recover the $T = 0$ critical value of Δ for the large-M component SG of quantum rotors with SK-type interactions [25].

Appendix B. The classical limit

In the limit of the vanishing of the parameter Δ, quantum effects are expected to become irrelevant and we should recover the classical solution for the spherical model [24]. In this limit, the path integral over positions and 'momenta' factorizes giving

$$Z \xrightarrow{\text{classical limit}} \int_{-\infty}^{+\infty} \prod_i \frac{\mathrm{d}\Pi_i}{2\pi}\, \mathrm{e}^{-\beta\mathcal{T}(\Pi)} \int_{-\infty}^{+\infty} \prod_i \mathrm{d}\sigma_i\, \delta\left(\sum_{i=1}^{N} \sigma_i^2(\tau) - N\right)\mathrm{e}^{-\beta V(\sigma)}$$
$$= \frac{1}{\sqrt{2\pi\beta\Delta}} Z_{\mathrm{cl}} \qquad \text{(B.1)}$$

where Z_{cl} is denotes the classical statistical sum of the classical spherical SG model [24]:

$$Z_{\mathrm{cl}} = \int_{-\infty}^{+\infty} \prod_i \mathrm{d}\sigma_i\, \delta\left(\sum_{i=1}^{N} \sigma_i^2(\tau) - N\right)\mathrm{e}^{-\beta V(\sigma)} \qquad \text{(B.2)}$$

and

$$\mathcal{T}(\Pi) = \frac{1}{2\Delta}\sum_i \left(\frac{\partial\sigma_i}{\partial\tau}\right)^2$$
$$V(\sigma) = -\sum_{i<j} J_{ij}\sigma_i\sigma_j. \tag{B.3}$$

From the relation (B.1) it follows that the classical free energy of the classical spherical model should be recovered from the limit

$$f_{\mathrm{cl}} = \lim_{\Delta\to 0}\left[f(\Delta, v_0) - \frac{1}{\beta}\ln(\sqrt{2\pi\beta\Delta})\right] \tag{B.4}$$

where $f(\Delta, v_0)$ is the free energy for the quantum spherical SG model on the Bethe lattice (37). Explicitly one has

$$f_{\mathrm{cl}} = \frac{1}{\beta}\left[\frac{1}{2}\int_{-J_c}^{J_c}\rho(\kappa)\ln\left(w_0 - \frac{1}{2}\beta\kappa\right)\mathrm{d}\kappa - w_0 - \frac{1}{2}\ln\pi\right] \tag{B.5}$$

where

$$w_0 = \lim_{\Delta\to 0}\beta v_0(\beta, \Delta). \tag{B.6}$$

The constraint equation is

$$1 = \frac{2(z-1)\left(2w_0(z-2) - \sqrt{4w_0^2 - (\beta J_c)^2}z\right)}{\beta^2 J_c^2 z^2 - 16w_0^2 z + 16w_0^2} \tag{B.7}$$

$$w_0 = \begin{cases} \dfrac{\left(\sqrt{4\beta^2 J^2 + 1} - 1\right)z}{4} + \dfrac{1}{2} & \text{for } k_B T > J\sqrt{z-2} \\ \beta J\sqrt{z-1} & \text{for } k_B T < J\sqrt{z-2} \end{cases} \tag{B.8}$$

so we can identify the critical temperature $k_B T_c = J\sqrt{z-2}$. Specializing to the classical limit $(\Delta \to 0)$ we determine

$$q_{\mathrm{EA}}(T, \Delta = 0) = 1 - \frac{\sqrt{z-1}}{z-2}\frac{k_B T}{J} \tag{B.9}$$

which vanishes at the classical critical temperature $k_B T/J = (z-2)/\sqrt{z-1}$. The local susceptibility in turn becomes

$$\chi_{\mathrm{cl}} = \begin{cases} \dfrac{\sqrt{z-1}}{J(z-2)} & \text{for } T < T_c \\ \dfrac{1}{k_B T} & \text{for } T > T_c. \end{cases} \tag{B.10}$$

Finally for the free energy we obtain

$$\begin{aligned} f_{\mathrm{cl}} = &-\frac{(z-2)}{8\beta}\ln|\beta^2 J_c^2 z^2 - 16w_0^2 z + 16w_0^2| - \frac{w_0}{\beta} - \frac{\ln\pi}{2\beta} + \frac{(z-2)}{8\beta} \\ &\times \ln\left[\frac{-z|\beta J_c \mathcal{S} z + 4w_0 z - 4w_0| + \beta J_c(z-2)^2\mathcal{S} + \sqrt{4w_0^2 - (\beta J_c)^2}\mathcal{Z}}{z|\beta J_c\mathcal{S} z - 4w_0 z + 4w_0| + \beta J_c(z-2)^2\mathcal{S} + \sqrt{4w_0^2 - (\beta J_c)^2}\mathcal{Z}}\right] \\ &- \frac{(z-2)}{8\beta}\ln\left(\frac{|\beta J_c\sqrt{z-1}z + 4w_0 z - 4w_0|}{|\beta J_c\sqrt{z-1}z - 4w_0 z + 4w_0|}\right) \\ &- \frac{z}{4\beta}\left[\ln(z-1) - \ln\left(\sqrt{4w_0^2 - (\beta J_c)^2} + 2w_0\right) + 3\ln 2\right] \end{aligned} \tag{B.11}$$

where, in the second line, we have used

$$\mathcal{S} = \sqrt{z-1} \qquad \text{and} \qquad \mathcal{Z} = (2z^2 - 6z + 4)$$

(simply to improve the layout). The resulting internal energy is

$$U = \frac{1}{2\beta} - J\sqrt{z-1} \tag{B.12}$$

whereas the specific heat is constant, $C = k_B/2$, within the glass phase and the entropy diverges logarithmically:

$$S = -\frac{k_B}{2}\ln(T/J).$$

However, this is not unexpected, since the classical spherical spin-glass model displays the same pathology [24].

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